Luciano Boggio

Università Cattolica del S.Cuore

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Growth unemployment and wages. Disequilibrium models with increasing returns.

Increasing returns are introduced in models of cyclical growth inspired by Goodwin's predator-prey model and in the neoclassical version produced by Akerlof and Stiglitz (1969). In this way, the technology of the latter becomes similar to that adopted in some endogenous growth models. As a result, the modified Goodwin model becomes unstable, while its neoclassical counterpart remains stable.

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Some major features of the dynamic interaction between growth and functional income distribution are captured in a very simple way by those models that – as in Goodwin (1969) – express it in two equations of the following type:

1) The growth rate of capital is inversely proportional to the income share going to the wage earners;

2) The growth rate of real wages is directly proportional to the share of employment in the labour force, which in turn depends on the amount of capital.

As is well known, the solution of Goodwin' s model is an oscillating trajectory around a steady state path associated with the natural rate of growth[1]. In our opinion, one of the merit of this model is that – contrary to both neoclassical and neokeynesian growth models – the full employment of the labour force is not taken for granted. This feature is due to the fact that the two above equations embodies an adjustment mechanism of the labour market.

Such adjustment mechanism – that here will be generalised to allow for a non-linear relationship between the growth rate of real wages and the rate of employment - when explicitly introduced in neoclassical and neokeynesian growth models, gives rise to different outcomes . In the former, stability of the steady state path associated with the natural rate of growth prevails, though in general this path is not of full employment (see Akerlof and Stiglitz (1969)). In the latter, the typical behaviour of Goodwin's model is preserved. In sections I,1 and II,1 of the paper this results are expounded in detail.

The main purpose of this paper, however, is to introduce increasing returns in the context of the disequilibrium models just described.

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[1] Defined as "the rate of growth that keeps the unemployment rate constant". For this definition, cf. Boggio and Seravalli (2001).

This step is justified by the strong emphasis that increasing returns receive in both the neo-keynesian literature (following the work of Kaldor) and in the "endogenous growth theory" (which can be seen as a new development of the neoclassical approach). In particular, as we shall see, a well-known model of this strand of theory (Romer, 1986), is formalised – following Barro and Sala–i-Martin (1995) - by a Cobb-Douglas production function allowing for increasing returns. These aspects will be dealt with in section I,2 and II,2 of the paper.

The "neokeynesian" version.

I.1 Throughout the paper we shall keep the following notations and assumptions. The growth rate g_w [2]of the real wage rate w_t is a differentiable decreasing function H of the rate of unemployment,

$$g_w = H(1 - R_t) \qquad H'<0, \qquad (1a)$$

where $R_t = L_t/N_t$, N_t is labour supply and L_t is labour demand. To simplify the exposition, we assume that H is defined for the whole interval [0, 1]. Hence we can write the following assumptions

$$0<H(0),\ H(1)<0 \qquad (1b)$$

which imply a rest point U such that

$$0= H(U) \qquad (1c)$$

We further assume

$$H''>0, \text{ when } (1-R_t)<U \qquad (1d)$$

The graph of this function (see Figure 1) may be called "real-wage Phillips curve".

The growth rate g_K of the capital stock, K_t , is given by the saving propensity, s_t, divided by the capital-output ratio, $v_t \equiv K_t /Y_t$:

$$g_K = s_t/v_t \qquad (2)$$

Moreover:

[2] For the growth rate of a variable x, we shall normally use the notation g_x.

$\lambda>0$ is the rate of exogenous labour-augmenting technical progress;

$g_N = n$ $\qquad Y_t/L_t \equiv y_t$ $\qquad K_t/L_t \equiv k_t$

In this section we shall assume a fixed-coefficient technology and neokeynesian saving propensities, s_W and s_P (out of wages and out of profits, respectively), $s_W < s_P$. $D_t \equiv w_t/y_t$ and R_t will be our state variables. Then, noticing that

$g_L = g_Y - g_y = g_K - g_y$

and, under the present assumptions,

$g_K = [s_P + (s_W - s_P)D_t]/v$ $\qquad g_y = \lambda$

we get the following differential equation system:

$$g_D = H(1 - R_t) - \lambda \qquad (3a),$$

$$g_R = g_L - n = g_K - \lambda - n = [s_P + (s_W - s_P)D_t]/v - \lambda - n \qquad (3b)$$

To have an equilibrium path, we must assume

$$(n+\lambda)v < s_P \qquad (A1)$$

$$\lambda < H(0) \qquad (A2)$$

The equilibrium point (D^*, R^*) can be obtained as follows.

$$0 = s_P/v + (s_W - s_P)D_t/v - n - \lambda \quad \Rightarrow \quad D^* = [(n+\lambda)v - s_P]\,(s_W - s_P)^{-1} > 0.$$

From (3a) we get

$$0 = H(1 - R_t) - \lambda$$

$$1 - R^* = H^{-1}(\lambda)$$

well defined because of (A2)

$$R^* = -H^{-1}(\lambda) + 1 > 0$$

Notice that the equilibrium is not of full employment.

The Jacobian matrix of the differential equation system *3* at the equilibrium point is

$$\begin{bmatrix} 0, & -D^*H' \\ R^*(s_W - s_P)/v, & 0 \end{bmatrix}$$

Its two non-vanishing elements are of opposite sign. Hence the determinant is positive and the trace is null. The eigenvalues are imaginary.

It can be shown that the equilibrium is a centre[3]. In spite of the introduction of some non-linearities, the main feature of Goodwin's model, that is an oscillating trajectory around a steady state path associated with the natural rate of growth, is preserved.

I.2 A natural way to introduce increasing returns in a neokeynesian context is to use Verdoorn-Kaldor equation[4]:

$g_y = a + b g_Y, \quad a,b>0,\ b<1$

or, because of a fixed capital-output ratio,

$g_y = a + b g_K$

Replacing λ by the second member of this equation, we get:

$$g_D = H(1 - R_t) - a - b g_K = H(1 - R_t) - a - b[s_P + (s_W - s_P)D_t]/v \qquad (4a),$$

$$g_R = g_K - n - a - b g_K = (1-b)\,[s_P + (s_W - s_P)D_t]/v - n - a \qquad (4b)$$

Let (D^{**}, R^{**}) the equilibrium point.

Instead of (A1) and (A2) we assume, for similar reasons,

$$(n+\lambda)v(1-b)^{-1} < s_P \qquad (A3)$$

$$a + b g_K < H(0) \qquad (A4)$$

$$D^{**} = [(n+a)v(1-b)^{-1} - s_P]\,(s_W - s_P)^{-1} > 0$$

$$R^{**} = -H^{-1}\{ a + b[s_P + (s_W - s_P)D^{**}]/v\} + 1 > 0$$

[5]and the Jacobian matrix becomes

$$\begin{bmatrix} -D^{**}\, b(s_W - s_P)/v, & -D^{**}H' \\ R^{**}(1-b)\,(s_W - s_P)/v, & 0 \end{bmatrix}$$

The determinant and the trace are both positive. The real part of both eigenvalues is positive and the equilibrium is unstable. The assumption of increasing returns in the form of a Kaldor –Verdoorn equation introduces a strong element of instability, by superimposing to the constant oscillation pattern a positive feed-back of D_t on itself.

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[3] See Boggio (2003).

[4] For a recent assessment of its theoretical and empirical foundation, see McCombie Pugno and Soro (2002).

[5] Again the equilibrium is not of full employment.

Suppose, for instance, that D_t, the share of wages, be higher than its equilibrium value. Then g_K and g_y will be lower than their equilibrium value, hence g_D will be higher and, barring cross-effects from R_t , D_t will move further away from its equilibrium value.

II. A "neoclassical" version.

II.1 To have a tractable model with both neoclassical factor substitution and increasing returns we choose a version of Romer (1986) given by Barro and Sala –i-Martin (1995, pp. 146-151).

In this model the effect of *learning-by-doing* and *technological spillovers* is captured by introducing in each firm's production function – a Cobb-Douglas – the total capital of the economy, which plays the role of labour augmenting technical progress. More precisely

$$Y_{jt} = AK_{jt}^{\alpha}(K_t L_{jt})^{1-\alpha} \qquad (5)$$

where suffix j denotes a single firm's variable, $j= 1, 2 , \dots m$.

A comparison with the model and results of section I is made difficult by the fact that, with a Cobb-Douglas technology and the usual neoclassical assumptions about factor rewards[6], the share of wage w_t/y_t is always equal to $(1-\alpha)$, hence one of the differential equation of the systems of section I breaks down. Therefore we are bound to adopt a different set-up[7]. As a first step we shall apply it to the more traditional case where the labour augmenting technical progress is exogenous.

We consider the following production function for the economy:

$$Y_t = AK_t^{\alpha}(L_t e^{\lambda t})^{1-\alpha}$$

and define the following auxiliary variables:

$$u_t \equiv w_t\, e^{-\lambda t} \qquad z_t \equiv K_t/N_t e^{\lambda t} \qquad x_t \equiv K_t/ L_t e^{\lambda t}$$

Then

[6] Which turn out to be necessary to determine, e.g., the demand for labour.

[7] Which generalises that of Akerlof and Stiglitz (1969).

$Y_t/K_t = Ax_t^{\alpha-1}$ $\quad\quad$ $v_t = x_t^{1-\alpha}/A$

Equality between the marginal productivity of labour and the wage rate gives:

$w_t = A\,(1-\alpha)e^{\lambda t}\,x_t^{\alpha}$ $\quad\quad$ $u_t = A\,(1-\alpha)\,x_t^{\alpha}$

$x_t = V(u_t) \equiv [\,u_t/A(1-\alpha)\,]^{1/\alpha},\ V'>0$

Hence, assuming a fix saving propensity *s,*

$$g_u = g_w - \lambda = H[1 - z_t/V(u_t)] - \lambda \quad (6a),$$

$$g_z = g_K - n - \lambda = s/v_t - n - \lambda = s\,A[\,V(u_t)]^{\alpha-1} - n - \lambda \quad (6b)$$

It can be shown that with this set-up a model with fixed coefficient technology and neokeynesian saving propensity would give an equilibrium point and a Jacobian matrix with the same qualitative properties as system (3).

The equilibrium point $(u°, z°)$ of system *(6)* can be obtained as follows.

$$s\,A\,[\,V(u_t)]^{\alpha-1} = n+\lambda \quad \Rightarrow [sA/(n+\lambda)]^{1/(\alpha-1)} = V(u_t)$$

$$\Rightarrow u° = V^{-1}\{[sA/(n+\lambda)]^{1/(\alpha-1)}\} > 0$$

$$H[1 - z_t/V(u°)] = \lambda \quad \Rightarrow -z°/V(u°) = [H^{-1}(\lambda) - 1] \quad \Rightarrow z° = -[H^{-1}(\lambda)-1]\,V(u°) > 0$$

Using again assumption (A2) we have an admissible solution.

The Jacobian matrix of differential system *(6)* at the equilibrium point is

$$\begin{bmatrix} u°H'z°V'/[V(u°)]^2, & -u°H'/[\,V(u_0)] \\ z°\,s\,A(\alpha-1)[\,V(u°)]^{\alpha-2}V', & 0 \end{bmatrix}$$

Since the determinant is positive and the trace is negative, the real part of both eigenvalues is negative and the equilibrium is asymptotically stable. The flexibility of the coefficients smoothes down the oscillatory pattern generated by the Goodwin-type feedbacks between wages and capital growth.

II.2 Let us now adopt the technology described at the beginning of this section, namely

$$Y_{jt} = AK_{jt}^{\alpha}(K_tL_{jt})^{1-\alpha} \quad (5)$$

where suffix *j* denotes a single firm's variable*, j= 1, 2 , …m*. Then, assuming that all firms are identical

$Y_t = \sum_{j=1}^{m} Y_{jt} = AK_t^{\alpha}(K_tL_t)^{1-\alpha} = A\, K_t L_t^{1-\alpha}$ (7)

and

$g_K = s\, A\, K_t L_t^{1-\alpha} / K_t = sA\, L_t^{1-\alpha}$ (8)

Equality between marginal productivity and wage rate gives:

$w_t = A\,(1-\alpha)k_{jt}^{\alpha} K_t^{1-\alpha} = A\,(1-\alpha)k_t^{\alpha} K_t^{1-\alpha}$ (9)

Now we want to study existence and stability of a constant-rate-of-growth path of K_t and w_t, in presence of a constant supply of labour that we fix equal to 1. Notice that such a path – that we shall call "equilibrium path" – implies $L_t = L$ a constant. Hence along this path, since

$Y_t / L = A\, K_t L^{-\alpha}$

$g_Y = g_y = g_K$

Moreover the constancy of w_t/y_t when *(9)* holds implies also

$g_y = g_w \quad \Rightarrow g_K = g_w$

From $g_K = g_w$, using eqs. (1) - where L_t now may take the place of R_t since $N_t=1$ - and (8),we get

$H(1 – L_t)= sA\, L_t^{1-\alpha}$

The solution of this equation for $L_t \in [0, 1]$ is the required equilibrium value of labour demand.

A clearer view of the matter can be obtained (see Figures 1 and 2) by defining function Q by

$Q(L_t) \equiv H(1 – L_t)$

and, as a consequence,

$Q' > 0$

$Q(L_t) < 0$, all $L_t < (1-U)$

$Q(L_t) > 0$, all $L_t > (1-U)$

$Q'' > 0$, all $L_t > (1-U)$

We need also the following function:

$M(L_t) \equiv sA\, L_t^{1-\alpha} - Q(L_t)$

Hence

$M' > 0$

$M(L_t) > 0$, all $L_t < (1-U)$

We add also the assumption[8]

$$M(1) = sA - Q(1) < 0 \qquad (A5)$$

Under these assumptions the equation $M(L_t)=0$ has one and only one solution, denoted by L_*. Along the equilibrium path,

$$g_K = sA\, L_*^{1-\alpha} = g_w = H(1 - L_*)$$

Again this equilibrium is not of full employment.

To study the stability of this path, let us get from eq. (9)

$$w_t = A\,(1-\alpha) L_t^{-\alpha} K_t \qquad (10)$$

and

$$L_t = [\,A\,(1-\alpha)\, K_t / w_t]^{\,1/\alpha}$$

$$g_L = \frac{1}{\alpha}\,(g_K - g_w) = \frac{1}{\alpha}\, M(L_t) \qquad (11)$$

By this equation we obtain a remarkable simplification of the analysis, which can now proceed with a single equation in one state variable!

Since (see Figure 2)

for all $L_t > L_*$, $M(L_t) < 0 \quad \Rightarrow g_L < 0$

for all $L_t < L_*$, $M(L_t) > 0 \quad \Rightarrow g_L > 0$

L_* is a globally stable equilibrium.

The same holds for the equilibrium path of K_t and w_t associated with L_*. On this path $g_Y = g_y = g_K = g_w$, $g_K = sA\, L_*^{1-\alpha}$, $g_w = H(1 - L_*)$ and

$K_t / w_t = 1/\, A\,(1-\alpha)\, L_*^{\alpha}$.

This strong stability result can be "explained" verbally as follows.

The growth rate of capital, $sA\, L_t^{1-\alpha}$, has a positive effect on the growth rate of L_t, while the growth rate of wages, $Q(L_t)$, has a negative effect. Since both are positively

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[8] This assumption is logically equivalent to (A2) and (A4): is necessary to ensure the existence of an equilibrium.

affected by the level of L_t, the latter is stabilising, the former is de-stabilising. Given the assumptions on the shape of the functions, the latter turn out to be dominant (see Figure 2).

Concluding remarks.

The stability of the neoclassical models – under both constant and increasing returns – supports the idea of a self-regulating growth process. More precisely, according to these models, the interplay between the growth of wages and that of capital, thanks to the stabilising effect of the flexibility of the coefficients, tends to restore the equilibrium condition, that is a steady state path, where national income grows at the natural rate, but there is no full employment.

The instability of the neokeynesian model with increasing returns implies upwards and downwards cumulative processes, moving away from the equilibrium path. A sufficiently regular growth then would require an active economic policy to smooth them out[9].

We like to stress that the ways increasing returns are introduced in the two models are not formally equivalent. To see this, from eq. (7) we derive

$y_t = A\, K_t L_t^{-\alpha} \qquad \Rightarrow g_y = g_K - \alpha\, g_L$

whilst our version of Kaldor-Verdoorn equation is

$g_y = a + b g_K$

However, each of these two ways of introducing increasing returns does not appear suitable for being introduced in <u>both</u> models, nor seems to exist another way more suitable for such a role, hence for a closer comparison of the two models with increasing returns.

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[9] In any case these models, which take for granted the equality between full-capacity desired saving and investment, implicitly assume the presence of some form of macroeconomic control of the economy.

Clearly the judgement between the relative merit of the two models and their implications - if not made on the basis of political or ideological *a priori* - should depend on one's opinion about the crucial assumptions, in particular about the flexibility of the production coefficients.

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Figure 1

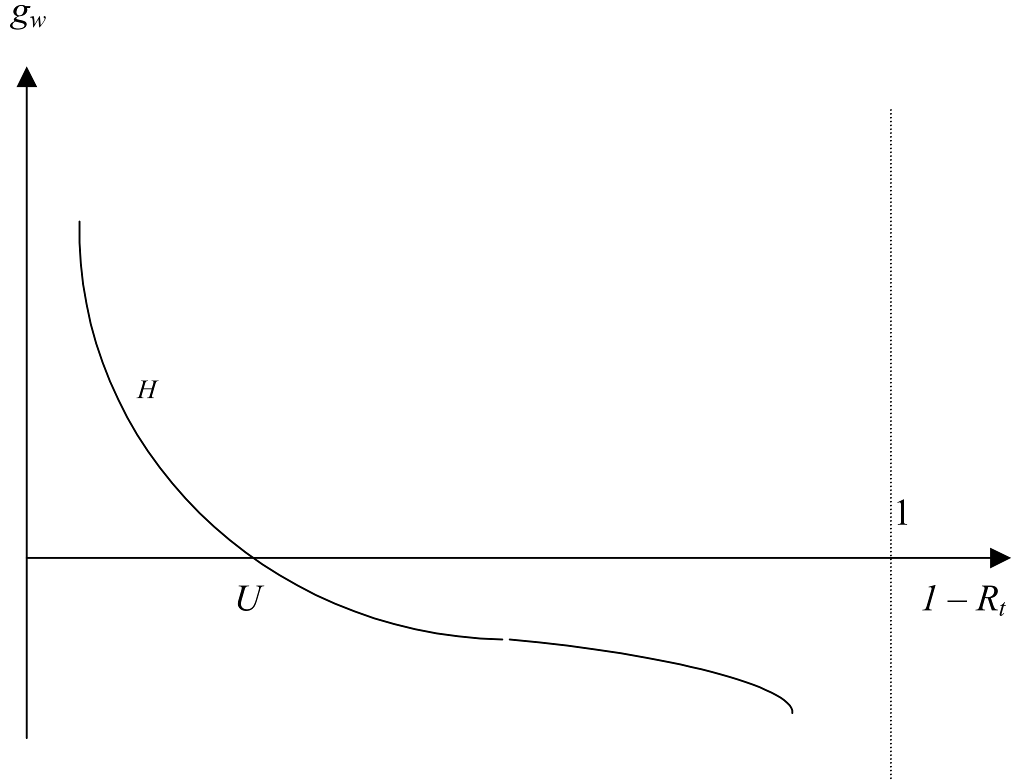

Figure 2