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Growth unemployment and wages. Disequilibrium models with increasing returns.

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The role of increasing returns in fostering economic growth, that was so prominent in Adam Smith's *Wealth of Nations*, has been rediscovered in recent years. Two names in particular deserve to be mentioned in this connection: within endogenous growth theory, that of Paul Romer[1]; within non-mainstream growth theory, that of Nicholas Kaldor[2].

In describing the elements generating increasing returns, contemporary theory has developed some of Smith's intuitions, using clarifications and deeper insights of later authors. This holds for learning by doing, thanks to Arrow's (1962) well-known paper and for the link between accumulation of capital and technical progress, which in Kaldor's (1957) technical progress function, become two hardly distinguishable aspects of the same process. Another important development is Young's (1928) interpretation of increasing differentiation and specialisation, which recently has attracted great interest and received important applications[3], particularly where Young stresses that, as the size of the economy grows,

> "[n]ew products are appearing, firms are assuming new tasks, and new industries are coming into being."(*ibidem*, p. 528)

Of course, one should also mention as an important clarification the Marshallian distinction between internal and external economies. It must be stressed, however, that today the role of the former is not undisputed. According to Romer (1991, p.103), in the very long run

> "[…T]he true cost function in terms of all rival inputs must exibit constant cost. […] If there are nonrival inputs [knowledge] as well, there must be a departure from homogeneity of degree 1."

Hence in the very long run economies of large scale should be ruled out. On the other hand, Romer's contextual remark that new knowledge is produced either intentionally

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[1] Cf. Romer (1986, 1987, 1990, 1991).

[2] Cf. in particular Kaldor (1966, 1967, 1981).

[3] Kaldor (1966 and 1967); Romer (1987 and 1990). For a comparison of Romer's (1987) and Young's approaches, cf. Lavezzi (2003)

or as a side effect of economic activity (*ibidem*, pp. 106-107) lead us to notice an element – intentional production of new knowledge – that traditionally had not been explicitly considered among the causes of increasing returns.

Increasing returns are an essential ingredient of several models of endogenous growth and Romer's view can be taken as representative of the way that theory sees them.

As for Kaldor (1966, 1972), whose growth theory was not meant for "the very long run", he included as factors causing increasing returns – along with learning by doing, increasing differentiation and specialisation in the way developed by Young - the economies of large-scale production and endogenous technical progress, to the extent that it is a side-effect of investment or concomitant with it. Then he tried to give to the notion of increasing returns an empirical counterpart, by proposing and quantifying a relationship – baptised "Verdoorn Law" [4] [5]- between the growth rate of output and that of productivity.

In what follows, to try to assess the role of increasing returns in the stability of steady state paths, when labour force, capital and functional income distribution are not on their equilibrium path, we shall use a model of endogenous growth based on increasing returns and one incorporating Verdoorn Law.

To this end, we shall make use of some pre-existing structures of out-of-equilibrium dynamics, linking those variables.

In our opinion, some major features of the dynamic interaction between growth and functional income distribution are captured in a very simple way by those models that – as in Goodwin (1967) – express it in two relationships of the following type:

1) The growth rate of capital is inversely proportional to the income share going to the wage earners;

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[4] After the name of the Dutch economist who studied this relationship several years before Kaldor (Verdoorn, 1949).

[5] For a recent assessment of its theoretical and empirical foundation, see McCombie Pugno and Soro (2002).

2) The growth rate of real wages is directly proportional to the share of employment in the labour force, which in turn depends on the amount of capital.

As is well known, the solution of Goodwin' s model is an oscillating trajectory around a steady state path associated with the natural rate of growth[6]. In my opinion, one of the merit of this model is that – contrary to most growth models – the full employment of the labour force is not taken for granted. Moreover, even the equilibrium solutions need not be of full employment. This last feature is due to the fact that relationship 2 above embodies an adjustment mechanism of the labour market, by which full employment generates a strong contractual power of workers, hence very high increases in real wages.

The two models presented below will be partly based on Goodwin's disequilibrium equations. In the former Goodwin's model will be generalised to allow for a non-linear relationship between the growth rate of real wages and the rate of employment. Then increasing returns will be introduced in such a context by means of Verdoorn Law. This will be called "Goodwin-type" model.

In the latter, Goodwin's equation embodying relationship 1 above – in which the growth rate of capital is equal to the rate of profit - will be replaced by a standard neoclassical one, where changes in income distribution affect the growth rate of capital by altering the capital output/ratio. In this context, increasing returns will be introduced by means of a model, proposed by Barro and Sala–i-Martin (1995) and derived as an example from the seminal paper by Romer (1986) on increasing returns and long run growth. This model is based on a Cobb-Douglas production function, which, because of learning by doing and knowledge spillovers, exhibits increasing returns[7].

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[6] Defined as "the rate of growth that keeps the unemployment rate constant". For this definition, cf. Boggio and Seravalli (2001).

[7] "The creation of new knowledge by one firm is assumed to have a positive external effect on the production possibility of other firms because knowledge cannot be perfectly patented or kept secret." (Romer 1986, p.1003).

As a first step these models will be studied *without* increasing returns: in the former, the typical behaviour of Goodwin's model is preserved; in the latter, stability of the steady state path associated with the natural rate of growth prevails, though in general this path is not of full employment (see Akerlof and Stiglitz, 1969). In sections I,1 and II,1 of the paper these results are expounded in detail.

After the introduction of increasing returns, the former model becomes unstable. On the contrary, the neoclassical/endogenous-growth model with increasing returns that we examine remains stable. These aspects will be dealt with in section I,2 and II,2 of the paper.

It must be stressed from the beginning, that the results obtained in this paper must be considered as a very preliminary exploration of the role of increasing returns in the stability of steady growth paths.

Common notations and assumptions.

0. Throughout the paper we shall keep the following notations and assumptions. The growth rate g_w [8] of the real wage rate w_t is a differentiable decreasing function H of the rate of unemployment,

$g_w = H(1 - R_t)$ $\qquad H'<0,$ $\qquad$ *(1a)*

where $R_t = L_t/N_t$, N_t is labour supply and L_t is labour demand. To simplify the exposition, we assume that H is defined for the whole interval [0, 1]. Hence we can write the following assumptions

$H(0)>0, H(1)<0$ $\qquad$ *(1b)*

which imply a rest point U such that

$0= H(U)$ $\qquad$ *(1c)*

An important feature of this relationship is that, as full employment is approached the bargaining power of the workers becomes very strong , determining very high increases in real wages. This is expressed by $H''>0,$ when $(1-R_t)<U$ $\qquad$ *(1d)*

[8] For the growth rate of a variable x, we shall normally use the notation g_x.

Therefore we shall make assumptions such that no equilibrium can be of full employment.

The graph of this function (see Figure 1) may be called "real-wage Phillips curve".

The growth rate g_K of the capital stock, K_t , is given by the saving propensity, s_t, divided by the capital-output ratio, $v_t \equiv K_t / Y_t$:

$$g_K = s_t/v_t \qquad (2)$$

Moreover:

$\lambda>0$ is the rate of exogenous labour-augmenting technical progress;

$g_N = n>0$ $\qquad Y_t / L_t \equiv y_t$ $\qquad K_t/L_t \equiv k_t$

The Goodwin-type model.

I.1 In this section we shall assume a fixed-coefficient production function[9] and

$s=(1–D_t)$, $\qquad D_t \equiv w_t/y_t$

$(1–D_t)$ is the income share of profits. D_t and R_t will be our state variables. Then, noticing that

$g_L = g_Y - g_y = g_K - \lambda$

$g_K=(1–D_t)/v$

we get the following differential equation system:

$$g_D = H(1 - R_t) - \lambda \qquad (3a),$$

$$g_R = g_L - n = g_K - \lambda - n = (1–D_t)/v - \lambda - n \qquad (3b)$$

[9] By fixed-coefficient production function we mean $Y_t = Min[AK_t; BL_t]$, where A and B are parameters that may vary over time for prescribed causes, but not for changes in the price of factors.

To have a meaningful equilibrium path, we must assume

$(n+\lambda) < 1/v$ (A1)

$\lambda < H(0)$ (A2)

The latter assumption can be viewed as a consequence of assumption (*1d)*, namely that the slope and the value of function *H* – hence the rate of growth of wages - become very large as full employment is approached.

The equilibrium point (D^*, R^*) can be obtained as follows.

From (3a) we get

$0=H(1-R_t)-\lambda$

$1-R^*=H^{-1}(\lambda)$

Because of (A2) $(1-R^*)$ is well-defined, positive and less than 1 (see Figure 1) and

$R^*=-H^{-1}(\lambda)+1$

positive and less than 1.

From (3b) we get

$D^*=1-(n+\lambda)v$

positive and less than 1.

Notice that the equilibrium is not of full employment.

The Jacobian matrix of the differential equation system (*3)* at the equilibrium point is

$$\begin{bmatrix} 0 & -D^*H' \\ -R^*/v & 0 \end{bmatrix}$$

Its two non-vanishing elements are of opposite sign. Hence the determinant is positive and the trace is null. The eigenvalues are imaginary.

It can be shown[10] that the equilibrium is a centre[11], generating over time a constant oscillation trajectory around a steady state path associated with the natural rate of growth. In spite of the introduction of a non-linearity in the right-hand side of (*3a)*, the main feature of Goodwin's model is preserved.

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[10].See the Appendix, *below.*

[11] This means that any trajectory in the phase space (D_t,R_t) is a closed orbit.

I.2 We introduce increasing returns in this context by means of a Verdoorn equation:

$g_y = a + bg_{Y},$ $a, b>0,$ $b<1$

A typical fixed-coefficient production function is not consistent with this equation and must be replaced by

$Y_t = Min[K_t/v;\ \Psi L_t^{\sigma}],$

where v and Ψ are positive parameters and $\sigma>1$ [12] .

Then, because of a fixed capital-output ratio,

$g_y = a + bg_K$

Replacing λ by the second member of this equation, we get:

$g_D = H(1 - R_t) -a-b(1-D_t)/v$ (4a),

$g_R = (1-b)\ (1-D_t)/v - n- a$ (4b)

Let us call *(D**, R**)* the equilibrium point.

Instead of (A1) and (A2) we assume, for similar reasons, (A3) and (A4):

$(n+a) <(1-b)/v$ (A3)

hence

$D^{**}=1-[(n+a)v(1-b)^{-1}]$

positive and less than 1;

$a+b(1-D^{**})/v <H(0)$ (A4)

The latter means that the equilibrium rate of growth of productivity cannot exceed the full-employment rate of growth of real wages.

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12 Then, assuming an efficient use of factors, $Y=\Psi L^{\sigma}$, so that $L=\left(\frac{Y}{\psi}\right)^{1/\sigma}$ and $y=\frac{Y}{L}=\psi^{1/\sigma}.Y^{1-1/\sigma}$ By multiplying the last member of this equation by e^{at} and setting $b= 1-1/\sigma$, one can easily derive the Verdoorn equation. See Boggio and Seravalli, 2003, pp.226-9.

$R^{**} = - H^{-1}\{ a + b\,(1–D^{**})/v\} + 1$

well-defined, positive and less than 1.

*Notice that increasing returns, as measured by the "Verdoorn coefficient" b, have a positive effect on the equilibrium growth rate of capital (1–D**)/v:* $\frac{\partial g_K}{\partial b} = \frac{n+a}{(1-b)^2} > 0$

The Jacobian matrix of the differential equation system 4 at the equilibrium point becomes

$$\begin{bmatrix} D^{**} b/v & -D^{**}H' \\ R^{**}(b-1)/v & 0 \end{bmatrix}$$

The determinant and the trace are both positive. The real part of both eigenvalues is positive and the equilibrium is unstable.

The assumption of increasing returns in the form of a Verdoorn equation introduces a strong element of instability, by superimposing to the constant oscillation pattern a positive feed-back of D_t *on itself.* Suppose, for instance, that D_t, the share of wages, be higher than its equilibrium value D^{**}. Then g_K and g_y will be lower than their equilibrium value, hence, if R_t is sufficiently close to R^{**}, g_D will be positive and D_t will move further away from D^{**}.

II. A neoclassical model.

II.1 To have a tractable model with both neoclassical factor substitution and increasing returns we choose a model[13] in which the effect of *learning-by-doing* and *technological spillovers* is captured by introducing in each firm's production function – a Cobb-Douglas – the total capital of the economy, which plays the role of labour augmenting technical progress.

More precisely

$$Y_{jt} = AK_{jt}^{\alpha}(K_t L_{jt})^{1-\alpha} \tag{5}$$

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[13] Derived as an example by Barro and Sala–i-Martin (1995, pp. 146-151) from Romer (1986).

where suffix j denotes a single firm's variable, $j= 1, 2, ...m$.

A comparison with the model and results of section I is made difficult by the fact that, with a Cobb-Douglas technology and the usual neoclassical assumptions about factor rewards[14], the share of wages w_t/y_t is always equal to $(1-\alpha)$, hence one of the differential equation of the systems of section I breaks down. Therefore we are bound to adopt a different set-up[15]. As a first step we shall apply it to the more traditional case where the labour augmenting technical progress is exogenous.

We consider the following production function for the economy:

$Y_t = AK_t^{\alpha}(L_t e^{\lambda t})^{1-\alpha}$

and define the following auxiliary variables:

the wage rate per efficiency unit (EU) $u_t \equiv w_t e^{-\lambda t}$

the capital/labour supply (in EU) ratio $z_t \equiv K_t/N_t e^{\lambda t}$

the capital/employment (in EU) ratio $x_t \equiv K_t/L_t e^{\lambda t}$

Then

$Y_t/K_t = Ax_t^{\alpha-1}$ $\quad v_t = x_t^{1-\alpha}/A$

Equality between the marginal productivity of labour and the wage rate gives:

$w_t = A(1-\alpha)e^{\lambda t} x_t^{\alpha}$ $\quad u_t = A(1-\alpha)x_t^{\alpha}$

$x_t = V(u_t) \equiv [u_t/A(1-\alpha)]^{1/\alpha}, \ V'>0$

The capital/employment (in EU) ratio depends on the wage rate per EU.

Hence, assuming a fix saving propensity s, we get a differential equation system in the state variables u and z:

$$g_u = g_w - \lambda = H[1 - z_t/V(u_t)] - \lambda \qquad (6a),$$

$$g_z = g_K - n - \lambda = s/v_t - n - \lambda = sA[V(u_t)]^{\alpha-1} - n - \lambda \qquad (6b)$$

The equilibrium point $(u°, z°)$ of system (6) can be obtained as follows.

[14] Which turn out to be necessary to determine, e.g., the demand for labour.

[15] Which generalises that of Akerlof and Stiglitz (1969).

$s A [V(u°)]^{\alpha-1} = n+\lambda \quad \Rightarrow [sA / (n+\lambda)]^{1/(1-\alpha)} = V(u°)$

$\Rightarrow u°=V^{-1}\{[sA/(n+\lambda)]^{1/(1-\alpha)}\}>0$

$H[1 - z°/ V(u°)]=\lambda \quad \Rightarrow \quad [1 - z°/ V(u°)] = H^{-1}(\lambda)$

By assuming again (A2), $H^{-1}(\lambda)$ is well-defined, $[1 - z°/ V(u°)]$ is positive and less than 1 (and the equilibrium is not of full employment). Hence

$z°=-[H^{-1}(\lambda)-1] V(u°)>0$

The Jacobian matrix of differential system *(6)* at the equilibrium point is

$$\begin{bmatrix} u°H'z°V'/[V(u°)]^2, & - u°H'/[V(u°)] \\ z° s A(\alpha-1)[V(u°)]^{\alpha-2}V', & 0 \end{bmatrix}$$

Since the determinant is positive and the trace is negative, the real part of both eigenvalues is negative and the equilibrium is asymptotically stable.

It can be shown that with this set-up a model with a fixed-coefficient production function and Goodwin-type saving propensity would have the same qualitative properties as our Goodwin-type model, that is a constant oscillation pattern. In these models the Jacobian matrix at the equilibrium point exibits a vanishing main diagonal and a positive determinant. Then, when the fixed-coefficient production function is replaced by a neoclassical one, the flexibility of the capital/labour coefficient with respect to wage changes generates in that matrix the negative upper-left element (appearing also in our present model). The real part of both eigenvalues becomes negative, smoothing down the oscillatory pattern.**In these models a neoclassical production function appears to be a strongly stabilising element.**

II.2 Let us now adopt the technology described at the beginning of this section, namely

$$Y_{jt}= AK_{jt}^{\alpha}(K_tL_{jt})^{1-\alpha} \quad (5)$$

where suffix j denotes a single firm's variable, $j= 1, 2 , ...m$. Then, assuming that all firms are identical

$$Y_t= \sum_{j=1}^{m} Y_{jt} = AK_t^{\alpha}(K_tL_t)^{1-\alpha} = A K_t L_t^{1-\alpha} \quad (7)$$

and

$g_K = s\,A\,K_t L_t^{1-\alpha} / K_t = sA\,L_t^{1-\alpha}$ (8)

Equality between marginal productivity and wage rate gives:

$w_t = A\,(1-\alpha)k_{jt}^{\alpha} K_t^{1-\alpha} = A\,(1-\alpha)k_t^{\alpha} K_t^{1-\alpha}$ (9)

$w_t = A\,(1-\alpha)L_t^{-\alpha} K_t$ (10)

hence

$L_t = [\,A\,(1-\alpha)\,K_t / w_t]^{1/\alpha}$

$g_L = \frac{1}{\alpha}\,(g_K - g_w)$ (11)

Let us assume a constant supply of labour that we fix equal to 1. Then equation (1) can be re-written as

$g_w = H(1 - L_t)$

and (11) as

$g_L = \frac{1}{\alpha}\,M(L_t)$ (12)

where

$M(L_t) \equiv sA\,L_t^{1-\alpha} - H(1 - L_t)$

By this equation we obtain a remarkable simplification of the analysis, which can now proceed with a single equation in one state variable!

The solution for $L_t \in [0,\ 1]$ of equation $M(L_t)=0$ is the equilibrium value of employment.

A clearer view of the matter can be obtained (see Figures 1 and 2) by defining function Q by

$Q(L_t) \equiv H(1 - L_t)$

and, as a consequence,

$Q' > 0$

$Q(1-U) = 0$

$Q'' > 0,$ all $L_t > (1-U)$

To discuss the properties of function $M(L_t)$, we notice that :

its first component $sA\,L_t^{1-a}$

is always positive;

its second component $-Q(L_t)$

changes its sign once , that is for $L_t \geq 1-U$ becomes negative.

Hence $M(L_t)$ changes its sign at most once.

We assume, as in Figures 2 and 3, that, as full employment is approached, the size and the slope of $-Q(L_t)$ become sufficiently large to grant that:

$M(L_t) = 0$ has a solution L_* and $M'(L_*)<0$ *(A5)*Along this equilibrium path,

$g_K = sA\, L_*^{1-\alpha} = g_w = H(1 - L_*)$

Again this equilibrium is not of full employment.

Since

for all $L_t > L_*$, $M(L_t)<0$ $\Rightarrow g_L<0$

for all $L_t < L_*$, $M(L_t)>0$ $\Rightarrow g_L>0$

L_* is a globally stable equilibrium.

The same holds for the equilibrium path of K_t and w_t associated with L_* . On this path $g_Y = g_y = g_K = g_w$ and

$K_t / w_t = L_*^{\alpha} / A\, (1-\alpha)$

This strong stability result can be "explained" verbally as follows.

The growth rate of capital, $sA\, L_t^{1-\alpha}$, has a positive effect on the growth rate of L_t, while the growth rate of wages, $Q(L_t)$, has a negative effect. Since both are positively affected by the level of L_t, the latter produces a negative, hence stabilising, feed-back; the former a positive, hence de-stabilising, feed-back. Given the assumptions that, as full employment is approached, wage increases become very large, the latter turns out to be dominant (see Figure 2).

Notice also that here *increasing returns are a very powerful source growth: productivity can grow at a constant rate, equal to that of capital.*

Concluding remarks.

One of the features of the models examined is that, contrary to most contemporary growth models, full employment is not consistent with steady state paths. As it was stressed before, this is the consequence of the assumption that, as full employment is approached, the bargaining power of the workers becomes very strong and wages grow faster than productivity. This assumption seems to me realistic [16]. It has also the merit of bringing in the role of socio-political and institutional factors. But in my opinion the main drawback of the assumption of full employment is that it prevents a proper stability analysis. This analysis, which is the focus of this paper, can hardly be meaningful, if it neglects labour market disequilibria and their effects on distribution.

However, some results of this paper seem to indicate that in the end that assumption is not really dangerous. The neoclassical models – under both constant and increasing returns – are stable: the interplay between the growth of wages and that of capital, thanks to the stabilising effect of the flexibility of the coefficients, tends to restore the equilibrium condition, that is growth along a steady state path. Therefore, according to these models, the assumption of full employment could be maintained without a real loss.

However, the fact that the Goodwin-type model becomes unstable when increasing returns are introduced, warn us that they are a potential source of instability: in that model, if the economy is not on a steady state, upwards and downwards cumulative processes set in, producing oscillations that move the economy away from the equilibrium path. A sufficiently regular growth then would require an active economic policy to smooth them out.

Notice also the ways increasing returns are introduced in the two models are not formally equivalent and they are stronger in the neoclassical model. To see this, let us recall that in this model along a steady state we have

$g_y = g_K$

[16] Of course, to assess its realism, full employment should be properly defined: not as absence of involuntary unemployment, but as equality between the number of vacancies and that of involuntary unemployed.

whilst our version of Kaldor-Verdoorn equation is

$g_y = a + bg_K, \qquad b<1$

Thus we may conclude that the instability potential of increasing returns is kept in check, when the stabilizing force of flexibility in coefficients is at work.

Of course, this require that the speed of adjustment of coefficients be high enough to accommodate the continuous shifts required by an out-of-equilibrium path.

But for those economists who – as the present writer – believe that, in any given state of technology, the scope for the flexibility of the capital/labour coefficient is rather limited and in any case this flexibility implies slow, difficult and uncertain processes, the stability of the neoclassical models looks a rather artificial result. Then increasing returns should appear as a powerful source of growth, but also, according to these very preliminary results, as a reason why, in order to ensure a sufficiently regular growth, the working of markets must be continuously supplemented by the action of economic policy.

However, much more work of investigation and richer models seem necessary to get less tentative and better founded results.

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Appendix: Non linear generalisation of Lotka-Volterra model.

Let f and g be two differentiable functions from R_+ to R, such that
$f'>0,\ f(y^*)=0$ and $g'<0,\ g(x^*)=0$, $(x^*,y^*)>0$.
We define the differential system

$$\frac{dx}{dt} = x\,f(y) \qquad (1a)$$

$$\frac{dy}{dt} = yg(x) \qquad (1b)$$

The Jacobian matrix of system *1* at (x^*,y^*) is

$$\begin{bmatrix} 0 & x^*f' \\ y^*g' & 0 \end{bmatrix}$$

Its two non-vanishing elements are of opposite sign. Hence the determinant is positive and the trace is null. The eigenvalues are imaginary: for the linearized system the equilibrium is a centre. Hence for the original system the equilibrium can be either a centre or a focus and a trajectory in the plane (x, y) can be either a closed orbit - in the former case – or - in the latter case - a spiral.
Let us show that the latter case cannot be true.
The derivative of the trajectory in the plane (y,x) is given by

$$\frac{dx}{dy} = \frac{xf(y)}{yg(x)} \qquad (2)$$

which implies

$$(g(x)/x)\,dx - (f(y)/y)\,dy=0 \qquad (3)$$

Therefore along a trajectory including point (A, B),

$$\int_A^x (g(x)/x)dx - \int_B^y (f(y)/y)dy = K$$

where K is a constant.

Hence letting

$P(x) \equiv \int_A^x (g(x)/x)dx,$

$Q(y) \equiv \int_B^y (f(y)/y)dy$

the implicit function of that trajectory is

$F(x, y) \equiv P(x)-Q(y)-K = 0$ (4)

The intersections of that trajectory with the horizontal line $\{x, y^*\}$ are positive solutions of the equation

$G(x) = 0$

where

$G(x) \equiv P(x)-Q(y^*)-K$

with

$G'(x) = P'(x) = g(x)/x$

Such intersections are at most two, since G' changes sign only once.

Hence the trajectory cannot be a spiral and the rest point of the system is a centre.

Figure 1

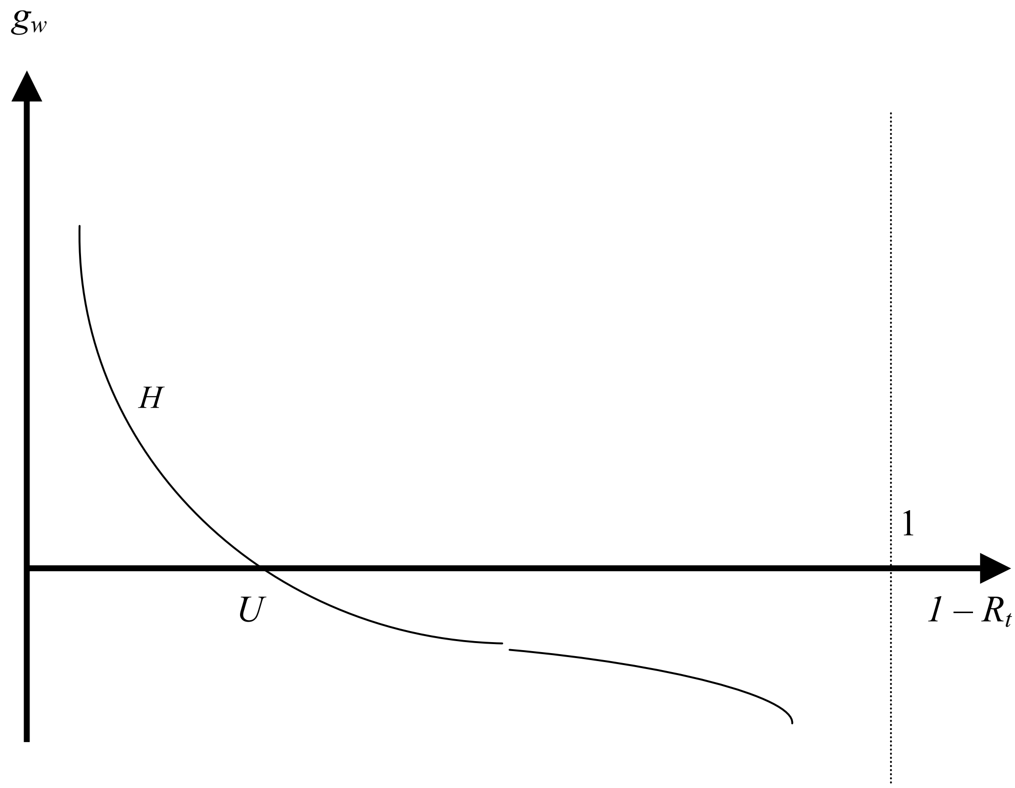

Figure 2