NATURAL WAGE DYNAMICS IN A RICARDIAN GROWTH MODEL

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> [Ricardo] retained *the important operative principle* that in any given social and cultural environment there is a "*natural* rate of wages" at which alone population could remain stationary and from which wages can only deviate temporarily. The hypothesis of an infinitely elastic supply curve of labour thus did not necessarily imply that this supply price must be equal to the bare minimum of subsistence. Yet this assumption was inconsistent with another (implied) feature of his model discussed below, that wages are not only *fixed* in terms of "corn" but are enterely (or almost entirely) *spent* on corn. (Kaldor 1956, p. 85 footnote 1, first emphasis is ours).

I. Introduction

In the last few decades a host of formal models have furnished Ricardo's theory of economic growth and income distribution with a mathematical garb.[2] We refer to Kaldor (1956), Samuelson (1959 and 1978), Pasinetti (1960), Hicks and Hollander (1977) and Casarosa (1978 and 1982), to mention just those which have commanded most attention. Extant Ricardian models share a common feature: they reproduce the classical distinction between a market rate of wages and a natural rate of wages and make it a crucial element to analyse the dynamic properties of the economy under study. According to Samuelson's 1978 canonical classical model, the rate of growth of labouring population, $(dL/dt)/L$, is an increasing function λ of the gap between the market rate of wages (w) and the natural rate of wages (w*), and a decreasing function of the sensibility of the growth of labouring population to the wage-gap (ε):

$\varepsilon(dL/dt)/L = \lambda(w - w^*)$; with $\lambda(0) = 0$, $\lambda'(.) > 0$ and $\varepsilon \geq 0$ (see Samuelson 1978, p. 1421, eq. 5).

Ricardian scholars differ as to their interpretation of Ricardo's view on the value of parameter ε. Some authors assume that for Ricardo the value of ε is zero: the Malthusian population mechanism works so rapidly that a growing economy may be analysed *as if* the market rate of wages were always at its historically determined natural level (see Pasinetti 1960, pp. 81 and 87). Accordingly, these authors grant a privileged position to the notion of natural wage in the analysis of the growth process.

By contrast, other authors assume that ε is large enough for population to grow "only slowly during a high-wage era" (Samuelson 1978, p. 1421). These authors generally make much of Ricardo's admission that "notwithstanding the tendency of wages to conform to their natural rate, their market rate may, in an improving society, *for an indefinite period*, be constantly above it" (*Works* I.v.94-95, emphasis added). Accordingly, they focus on market wage dynamics, with natural wage playing only a subsidiary role. In particular, for Casarosa a growing economy may be analysed *as if* the market rate of wages were always very close at its "dynamic equilibrium level" defined as the rate of wages at which "the rate of increase of population and the rate of capital accumulation are equal" (Casarosa 1978, p. 41).[3]

Our aim in this paper is not to assess the hits and the misses of the two major schools of thought on Ricardo's theory of wages in a growing economy. We rather concentrate on an analytical issue which both schools has not yet investigated in due detail. We refer to the relationship between economic growth and the secular dynamics of the natural rate of wages in real or commodity terms. Both schools assume that natural real wage is a given *and* constant magnitude. The assumption of constant natural real wage is an useful simplifying assumption which helps drawing many interesting growth results. Yet, models built on such an assumption do less than full justice to Ricardo's and, more generally, the classical point of view on the relationship between economic growth and the dynamics of natural real wage.[4] Ricardo explicitly warns his readers that "the natural price of labour, *estimated even in food and necessaries*, [is not to be understood as] absolutely fixed and constant. *It varies at different times in the same country*, and very materially differs in different countries" (*Works* I.v.96, emphases added). Smith, Torrens and Malthus make similar claims.

To put it in a nutshell, classical authors maintain that the amount and composition of workers' normal consumption basket depends on socio-political factors generally labelled as the "habits and customs" ruling in a given country in a given historical moment.[5] Classical economists consider habits and customs as persistent phenomena, once they are generally established among labouring population.[6] Nonetheless, classical authors do not treat habits and customs as purely exogenous magnitudes which fall outside the field of economic analysis. By contrast, classical economists hold that habits and customs are deeply influenced by economic factors, in particular by the past and present growth performance of a given country.[7] As a consequence, the historical evolution of habits and customs and the related dynamics of natural real wage are a fit subject for economic analysis.

Unfortunately, classical economists fail to explain in due detail the causal mechanism through which economic growth influences the dynamics of the natural real wage. Perhaps the clearest and most concise single statement of the classical point of view on the relationship between habits and customs, workers' normal consumption basket, natural real wage and stages of economic development may be found in Chapter I ('On the General Principles which Regulate Wages') Section 4 ('The Minimum of Wages') of Robert Torrens' *On Wages and Combinations*. The passage in question is worth of being fully quoted:

> The minimum below which wages cannot permanently fall, consists in a quantity of the necessaries and conveniences of life sufficient to preserve the labourer in working condition, and to induce him to keep up the race of labourers. The point, below which wages cannot fall, is not a fixed and immutable point, but is, on the contrary, liable to considerable variation. […] Even in countries, situate in the same climate, different habits of living will often occasion variations in the minimum of wages, as considerable as those which are produced by natural causes. The labourer in Ireland will rear a family under circumstances which would not only deter an English workman from marriage, but would force him on the parish for personal support. Now, it is certain, that a gradual introduction of capital into Ireland, accompanied by such a diffusion of instruction amongst the people, as would impart to them a taste for the comforts of life, might raise the minimum of wages in that country to an equality with their minimum in England […]. Alterations, however, in the minimum of wages cannot be suddenly effected. So far as this minimum […] is determined by the habits of living, and the established scale of comfort, it can be effected only by those circumstances of prosperity or decay, and by those moral causes of instruction and civilization, which are ever gradual in their operation. The minimum of wages, therefore, though it varies under different climates, and with the different stages of national improvement, may, in any given time and place, be regarded as very nearly stationary (Torrens 1834).

It is to be stressed that Torrens takes for granted that if Irish workers lived in a growing economy (such as England) instead of living in a stagnant economy (such as Ireland) they would learn to appreciate higher-quality, non-subsistence, commodities and they would start to control their fertility. Hence, in the light of Torrens' remarks above, we reconstruct the basic tenets of the classical point of view on the relationship between economic growth and natural wage dynamics as follows:

i) workers living in countries located in "different stages of national improvement" (such as England and Ireland up to the 1830s) turn out to develop different "habits of living" and different "established scale[s] of comfort". As a consequence, they earn different natural real wage (Torrens' "minimum of wages"),

ii) workers' "habits of living" and workers' "established scale of comfort" depend on the "circumstances of prosperity or decay".

To put it briefly, we claim that for classical economists, workers' normal consumption choices and workers' normal fertility choices are the two basic channels through which economic growth influences the dynamics of the natural real wage. On the one hand, workers earning 'high' real market wage acquire the economic possibility to buy higher-quality, non-subsistence commodities. On the other hand, once workers become aware of the trade-off between children to rear and "comforts of life" to enjoy, their fertility decisions cease being ruled merely by the "passion between the sexes" (to use Malthus' favourite expression) and start being disciplined by rational economic reasonings. Granted that the growth process goes on unimpeded and that the "moral causes of instruction and civilization" are at work, workers get progressively accustomed to higher standards of living and thus revise permanently their concept of subsistence both from a quantitative and a qualitative point of view. Natural real wage, slowly but steadily, rises.

Our argument unfolds in two stages. In the first stage (Section II of the paper) we inquiry into the historical background of classical analysis. We assess the compatibility of the classical economists' opinion on the rising trend of the natural real wage in eighteenth century England with the assumption of constant natural real wage. We propose as a solution to the puzzle the possibility that workers may take account, in their normal consumption and fertility choices, of some growth-induced changes. We refer to the movement of relative natural prices and the rise of workers' real income provoked by the process of economic growth. In the second stage (Section III of the paper) we propose more formal analysis. We elaborate a simple Ricardian growth model and we develop an extension of the model in order to analyse the dynamics of the natural rate of wages in the light of our findings in the first stage.

We are aware that a multi-commodity model is required in order to take full account of the impact of workers' consumption and fertility decisions on the dynamics of the natural wage in a Ricardian growing economy. Yet, we content ourselves in this paper with the analysis of a more simple one-commodity Ricardian growth model in which we drop the assumption of a constant natural real wage. (Needless to add, we postpone the analysis of natural wage dynamics in a multi-commodity framework to a further stage of our research.) In our model we show that a 'high' market wage may attract the natural wage. Making use of the language of physics, we claim that wage dynamics in a Ricardian growing economy may exhibit hysteresis. Apparently, such findings sound as a radical subversion of the received view on Ricardian economics which asserts that the market wage is attracted by the natural wage and not the other way round. As is well known, Ricardo maintains that "however much the market price of labour may deviate from its natural price, it has, like commodities, a tendency to conform to it" (*Works* I.v.94). Yet, what Ricardo calls

the "tendency" of the market wage towards the natural wage requires that workers' normal consumption basket be assumed as a given *and* constant magnitude. Say it differently, the Ricardian "tendency" of the market wage towards the natural wage cannot be taken for granted in economies where the growth process induces a drastic modification of the habits and customs which shape workers' normal consumption basket.

II. Historical Background: Classical economists on the dynamics of the natural rate of wages

Classical economists basically agree on defining the market rate of wages as a magnitude originated in the labour market and determined by the interplay of the supply of and the demand for common labour.[8] By contrast, classical economists propose a few definitions of the natural rate of wages which differ in many aspects but share a common feature: the natural rate of wages is *not* conceived as a directly observable magnitude (see Pasinetti 1982, p. 240).[9] As observable proxies for the natural rate of wages, classical economists choose to gaze at workers' normal patterns of consumption and fertility in different countries and in different historical periods. As far as eighteenth century England is concerned, they broadly agree that English workers

i) have increased their consumption of higher-quality commodities (both agricultural and manufactured ones) and

ii) have learnt to control their fertility in the boom years characterized by rapidly increasing demand for labour and a 'high' real market wage.

From observations i) and ii) above, classical economists conclude that the very concept of subsistence is drastically changed and that natural real wage in England has followed a markedly rising trend in the period under observation.[10] Such conclusion entails an interpretative puzzle in so far as it clashes with the twin Ricardian assumptions on natural real wage, that is to say,

1) the commodity composition of the natural wage-basket is given and constant and

2) the natural wage-basket is mostly made by subsistence, low-quality, agricultural commodities.

Ricardo's well-known prediction about the raising trend of natural nominal wage (and the related fall of the rates of profits and capital accumulation) is based on the twin assumptions above. According to Ricardo, in fact, "with the progress of society the natural price of labour has always a tendency to rise, because *one of the principal commodities by which its natural price is regulated* [food], has a tendency to become dearer, from the greater difficulty of producing it" (*Works* I.v.93,

emphasis added). Obviously, if "with the progress of society", food ceases to be "one of the principal commodities" which regulate the natural price of labour, nominal natural wage may not increase and the rates of profits and capital accumulation may not decrease.

In our view, the key to the puzzle above lies into workers' reaction to the wider consumption opportunities disclosed to them by the growth process. Growth in a Ricardian framework opens up workers the economic possibility of consuming a wage-basket different from the natural one determined by the historically given habits and customs. The natural real wage stays constant, *cæteris paribus*, if and only if workers do not modify their normal behaviour as regards consumption and fertility choices and thus they do not take advantage of the growth-induced new opportunities. As we shall see, this proposition holds both in a Ricardian economy where the market and the natural wage always coincide and in a Ricardian economy where the market wage is persistently above the natural wage.[11]

Consider first a Ricardian economy where the market and the natural wage always coincide. For Ricardo, workers' normal consumption basket includes both commodities produced by the agricultural sector of the economy ("food and necessaries") and commodities produced by the manufacturing sector of the economy ("conveniences"). In a closed economy without technical progress, agriculture is depicted by Ricardo as a sector whose technology displays decreasing returns to scale (the various qualities of land are in fixed supply), while manufactures are depicted as a sector whose technology displays increasing returns to scale (thanks to the Smithian process of division and specialization of labour). To put it briefly, in Ricardo's framework, agricultural commodities are increasing-price commodities; while manufactured commodities are decreasing-price commodities. Thus, in a Ricardian growing economy, relative natural prices are bound to change. Ricardo is perfectly aware of such natural prices movement. He writes that, in the course of economic growth, agricultural commodities have "a tendency to become dearer, from the greater difficulty of producing [them]"; while the natural prices of manufactured commodities "has a tendency to fall […] for though, on one hand, they are enhanced in real value, from the rise in the natural price of the raw material of which they are made, this is more than counterbalanced by the improvements in machinery, by the better division and distribution of labour, and by the increasing skill, both in science and art, of the producers" (*Works* I.v.93-94).

For Ricardo, the movement of relative natural prices, triggered by the process of economic growth, allows workers to carry on substitutions in consumption. Ricardo points out that from "manufactured commodities always falling, and raw produce always rising, with the progress of society, such a disproportion in their relative value is at length created, that in rich countries a

labourer, by the sacrifice of a very small quantity only of his food, is able to provide liberally for all his other wants" (*Works* I.v.97). Thus, even if the market rate of wages is always at its historically determined natural level, the growth-induced movement of relative natural prices allows workers to consume a wage-basket different from the one determined by the ruling habits and customs. Workers may thus revise their concept of subsistence, provided that they develop a taste for higher-quality, non-subsistence commodities.

Now consider a Ricardian economy where the market wage is persistently above the natural wage. The range of workers' consumption options is obviously wider than in the previous case since economic growth provokes not only a movement of relative natural prices but also a 'high' real income (estimated in terms of subsistence commodities). In such a scenario, it is highly probable that workers may change their normal consumption choices *and* their normal fertility choices. These two changes are not independent but may be seen as the two sides of the same coin.[12] Once a 'high' market rate of wages has opened up workers the possibility to achieve higher standards of living, workers may become progressively aware of the trade-off between the consumption of higher-quality commodities and the maintenance of a wider family. Hence the Malthusian population mechanism may collapse in the sense that population response to the wage-gap may decline: in terms of Samuelson's 1978 canonical classical model this means that the value of parameter ε may increase. If that is the case, labouring population keeps (almost) stationary in the face of a 'high' market rate of wages. Accordingly, the tendency of a growing economy to generate a 'high' market rate of wages is, *cæteris paribus*, strengthened. Since the 'high' market rate of wage does not provoke an increase of labouring population, the Ricardian natural/market wage distinction makes sense only by saying that the natural rate of wages has increased.[13]

Ricardo recognizes that "the increase of population, and the increase of food will generally be the effect, *but not the necessary* effect of high wages" (*Works* I.xxxii.406, emphasis added). Moreover, he is perfectly aware that the causal mechanism which links 'high' wages, workers' consumption decisions and population growth has a circular nature:

> The amended condition of the labourer, in consequence of the increased value which is paid him, does not necessarily oblige him to marry and take upon himself the charge of a family -he will, in all probability, employ a portion of his increased wages in furnishing himself abundantly with food and necessaries, -but with the remainder he may, if it please him, purchase any commodities that may contribute to his enjoyments—chairs, tables, and hardware; or better clothes, sugar, and tobacco. *His increased wages then will be attended with no other effect than an increased demand for some of those commodities; and as the race of labourers will not be materially increased, his wages will continue permanently high* (*Works* I.xxxii.406-407, emphasis added).

Malthus' analysis of the subject is more detailed. Malthus claims that economic growth has a direct influence on the formation and evolution of workers' habits of consumption and fertility thanks to the growth-induced possibility of buying higher-quality commodities:

> The condition of the labouring classes of society must evidently depend, partly upon the rate at which the funds for the maintenance of labour and the demand for labour are increasing; and partly, on the habits of the people in respect to their food, clothing, and lodging. […] It rarely happens, however, that either of them remains fixed for any great length of time together. […] *In general, their tendency is to change together*. When the funds for the maintenance of labour are rapidly increasing, and the labourer commands a large portion of necessaries, it is to be expected that *if he has the opportunity of exchanging his superfluous food for conveniences and comforts, he will acquire a taste for these conveniences, and his habits will be formed accordingly* (Malthus 1986, vol. V, pp. 182-183, emphases added).

As well as Ricardo, Malthus investigates the link between consumption choices and fertility choices in a growing economy characterized by 'high' market wages. His basic view is that a change of consumption habits may imply a change of fertility habits:

> From high real wages, or the power of commanding a large portion of the necessaries of life, two very different results may follow; one, that of a rapid increase of population, in which case the high wages are chiefly spent in the maintenance of large and frequent families; and the other, that of a *decided improvement in the modes of subsistence, and the conveniences and comforts enjoyed, without a proportionate acceleration in the rate of increase* (Malthus 1986, vol. V, p. 183, emphasis added).

Malthus' second result above comes true whenever workers become able and willing "to reason from the past to the future", and are not "ready to acquiesce, for the sake of present gratification, in a very low standard of comfort and respectability" (Malthus 1986, vol. V, p. 184). Once "the improvement in the modes of subsistence" becomes a persistent phenomenon, workers update, so to speak, their established standards of living: the rate of wages previously perceived as 'high' starts being considered as normal. That is the reason why Malthus criticizes Smith who, for Malthus, fails to appreciate the circular nature of the relationship between the natural rate of wages, on the one hand, and normal consumption and fertility choices, on the other hand. Smith establishes a one-way causal relationship between these two variables: he maintains that English workers normally consume better food than their Scottish counterparts because the natural wage in England is higher than in Scotland (see WN I.viii.32). Malthus' reply to Smith is that the above relationship may also work the other way round: for Malthus the natural wage in England is higher than in Scotland because English workers are accustomed to consume better food than their Scottish colleagues and thus they would refuse to marry and be obliged to consume the low-quality food

which Scottish workers normally accept to consume. Thus, for Malthus, the natural wage dynamics and workers' choices are strictly interrelated: 'high' wages induce a 'high' concept of subsistence and thus a 'low' rate of marriages and population increase which, at its turn, keeps wages at a 'high' level. By contrast, a 'low' concept of subsistence, generated by past 'low' wages, induces a 'high' rate of marriages and population increase which, at its turn, keeps wages at a 'low' level:

> *The effect, in this case as in many others, certainly becomes in its turn a cause*; and there is no doubt that if the continuance of low wages for some time, should produce among the labourers of any country habits of marrying with the prospect only of a mere subsistence, such habits, by supplying the quantity of labour required at a low rate [of wages], would become a constantly operating cause of low wages (Malthus 1986, vol. V, p. 183, emphasis added).

To sum up. Classical economists agree that economic growth may influence workers' consumption and fertility choices and thus may affect the secular dynamics of the natural real wage. Since economic growth widens the range of consumption options available to workers, the dynamics of the natural real wage depends on workers' decision to take advantage or not of the growth-induced opportunities to ameliorate their standards of living. Paraphrasing Malthus, it may be said that in a growing economy workers "are really the arbiters of their own destiny" (Malthus 1986, vol. V, p. 226).

III. Formal Analysis

In what follows we first sketch a canonical one-commodity Ricardian model in subsection III.1; then we extend the model to investigate market wage dynamics *and* the related dynamics of the natural wage in subsection III.2. Since our model admits multiple steady-state equilibria in terms of the wage rate and the level of the labour force, subsection III.3 discusses the general features of long-run equilibria and local stability. Finally, subsection III.4 proposes an analysis of the dynamics generated by the model when constant elasticity of substitution of labour is assumed.

III.1 The canonical Ricardian model

The canonical one-commodity Ricardian model describes a very simple agricultural economy with no foreign trade and no technical progress. The only commodity produced, 'corn', is produced by means of itself and labour (paid in corn). The amount and fertility of the various plots of land on which corn is cultivated are assumed as given and constant. Thus the corn-producing technology is represented by means of an equation of the type

$$X = f(N) \text{ with } f(0) = 0,\ f'(N) > 0,\ f'(\infty) < w^* < f'(0) \text{ and } f''(N) < 0 \tag{1}$$

where X is the amount of corn yearly produced by N labourers and w* is the natural rate of wages.

The analytical skeleton of the model also includes the following equations:

(2) $X = W + P + T$

representing the distribution of the national product X among aggregate Wages (W), Profits (P) and Rents (T);

(3) $T = f(N) - Nf'(N)$

representing the Ricardian theory of differential rents;

(4) $W = wN$

representing the Ricardian theory of wages where w is the ruling rate of wages;[14]

(5) $r \equiv P/K = P/W$,

which defines the rate of profits (r) as the ratio between aggregate Profits and aggregate Wages, K = W being the aggregate capital. (In a one-commodity framework like the one we are considering, capital is present only in the form of circulating capital and coincides with the total amount of anticipated wage-goods.)

Say's Law of Markets holds: aggregate income is entirely spent. Aggregate Profits are determined as a *residuum*, while the two distributive variables w and r are linked by a relationship of the type

(6) $wR = f'(N)$, with $R \equiv (1 + r)$

derived by eqq. (1)-(5).

The model thus reproduces the two fundamental propositions of the Ricardian theory of income distribution:

i) for each given level of N and $f'(.)$, the w-r relationship is an inverse one and

ii) as N increases, wR decreases. (This obviously implies that, if w is assumed as given and constant at the subsistence level, the law of diminishing returns causes a fall of r)

To make the model a dynamic one, we introduce three more equations. The laws of motion of capital and labouring population are represented by means of two equations of the type

(7) $g_N = \alpha((w - w^*)/w^*)$ with $\alpha(0) = 0$ and $\alpha'(.) \gtrless 0$

(8) $g_K = \beta(r - r^*)$ with $\beta(0) = 0$ and $\beta'(.) \gtrless 0$ for $r \geq r^*$

where g_K and g_N are the rate of growth of K and N. We define r* as the minimum rate of profits compatible with entrepreneurs' incentive to employ labour.[15] Here we follow Casarosa (1982) by assuming that below the minimum rate of profits r* there is no accumulation of capital. Hence $K \leq Nf'(N)/R^*$ with $R^* \equiv (1 + r^*)$.

The law of motion of the market wage is derived by the equation K = W = wN and represents labour market equilibrium:

(9) $g_w = g_K - g_N$

where g_w is the rate of growth of the market wage.

By contrast, the model assumes w* as a given *and* constant magnitude:

(10) $g_{w^*} = 0$

where g_{w^*} is the rate of growth of w*.

The model represented by equations (1) – (10) reproduces in a straightforward way the two fundamental propositions of the Ricardian theory of growth (see Casarosa 1982):

a) the economy's driving force is the accumulation of capital whose pace is determined by the state of income distribution, given entrepreneurs' propensity to accumulate, and

b) the dynamics of labouring population is endogenous to the model: given the Malthusian population mechanism the growth of capital creates, so to speak, the required growth of labour.

III.2 An extension of the model: the dynamics of the natural wage

The well-known Ricardian one-commodity model sketched above may be generalised in a number of directions (see Freni 1998). In this paper we investigate the dynamics generated by the model when eq. (10) is relaxed according to the suggestions of Ricardo and other classical economists. Thus we replace eq. (10) with eq. (10bis):

(10bis) $g_{w^*} = \gamma((w - w^*)/w^*)$ with $\gamma(0) = 0$ and $\gamma'(.) > 0$

We have already defined

(11) $R \equiv 1 + r$ and $R^* = 1 + r^*$

which represent a linear transformation of r and r*. It is now convenient to define a new variable:

(12) $w_D \equiv w/w^*$

which represents the wage normalized to its natural level.

It is straightforward from (6) that

(13) $g_R = [f''(N)N/f'(N)]g_N - g_w = -(1/\varepsilon(N)g_N - g_w$, for $R \geq R^*$

where $\theta(N) \equiv -f'(N)/f''(N)N \geq 0$ measures the elasticity of substitution of labour as defined by Hick and Hollander (1977, p. 358). In many economic applications such elasticity is constant and greater than 1. (Consider for example the exponential production function $f(N) = N^{\phi}$ with $\phi \in (0,1)$ where $\theta = 1/(1-\phi) > 1$). We assume that $d\theta/dN \leq 0$.[16] From eqq. (12)-(13) we have:

(14) $g_R = [1 - 1/\theta(N)]\,\alpha(w_D - 1) - \beta(R - R^*)$ for $R \geq R^*$.

The growth rate of w_D is given by:

$$g_{wD} = g_w - g_{w^*}$$

which implies

(15) $g_{wD} = \beta(R - R^*) - \alpha(w_D - 1) - \gamma(w_D - 1) = \beta(R - R^*) - \sigma(w_D - 1)$ for $R \geq R^*$

where $\sigma(w_D - 1) = \alpha(w_D - 1) + \gamma(w_D - 1)$ has the same characteristics of $\alpha(.)$ and $\gamma(.)$, that is to say, $\sigma(0) = 0$ and $\sigma'(.) > 0$.

Finally we trace the dynamics of N in terms of the new variable w_D:

(16) $g_N = \alpha(w_D - 1)$

Equations (15)-(17), together with the initial conditions

(17) $R(0) = f'(N_0)N_0/K_0$ and $w_D(0) = K_0/(w^*_0 N_0)$

fully describe the dynamics of our economy. (N_0, K_0 and w^*_0 are respectively the initial values of the labour force, the capital stock and the natural rate of wages.) In fact, any other variable, like capital and market wage, can be derived from the behaviour of w_D, N and R.

III.3 Long-run equilibrium and local stability

The dynamical system (14) – (17) has at least an equilibrium, *i.e.* a couple of R and w_D such that $g_R = g_{wD} = g_N = 0$ for $R^E = R^*$ and $w_D^E = 1$.[17] The features of such equilibrium are the usual ones: the rates of wages and profits are at their natural level while capital and labouring population are constant over time. However, we notice that the natural wage is endogenously determined and depends on the initial conditions of the economy. In the same manner, the equilibrium levels of capital and labouring population depend on the equilibrium level of the natural wage. To put it briefly, our model admits multiple steady-state equilibria in terms of the wage rate and the level of the labour force.

An interesting question to investigate is: which initial conditions can lead to an equilibrium characterized by higher long-run natural and market wages? To answer this question we first have to analyse the overall dynamics of the economy.

The first step in the analysis of stability is to check whether equilibrium $(R^E, {}_{wD}{}^E) = (R^*, 1)$ is locally stable. Thus we linearize around the steady state the dynamical system (14) – (17) and calculate the following eigen-values:[18]

$$(18) \quad \lambda_{1,2} = -(\sigma'(0) + \beta'(0)R^*)/2 \pm (1/2)[(\sigma'(0) - \beta'(0)R^*)^2 + 4\beta'(0)R^*\alpha'(0)(1 - 1/\theta(N^*))]^{1/2}$$

$$\lambda_3 = 0.$$

The presence of a zero eigen-value and two negative eigen-values (or with negative real part) makes the analysis not conclusive (see Gandolfo 1997, p. 362).[19] The origin of the problem is that the dynamics of N does not affect the other variables in the linearized system including itself; whereas N affects g_R in the original, non-linearized, dynamical system.

This fact suggests a useful simplifying assumption: to reduce the analysis to the (R, w_D) space.[20] Consider that if $\theta(N) = \theta$, then the dynamics of the economy is fully represented by the dynamics of R and w_D only. In this case $\lambda_{1,2}$ of eq. (18) with $\theta(N^*) = \theta$ represent the eigenvalues of the simplified dynamical system and therefore the system proves to be locally stable (the real part of both eigenvalues are negative). Moreover, under the following condition:

$$(19) \quad \theta \geq 4\beta'(0)\alpha'(0)\, R^* / (4R^*\beta'(0)\alpha'(0)\, R^* + (\sigma'(0) - \beta'(0)R^*)^2) = \underline{\theta} < 1$$

both eigenvalues are negative and real. Therefore the equilibrium is a stable node. Otherwise equilibrium $(R^E, {}_{wD}{}^E) = (R^*, 1)$ is a stable focus.[21]

III.4 Dynamics with constant elasticity of substitution of labour

In the previous subsection we have shown that, for a constant elasticity of substitution of labour, equilibrium $(R^E, {}_{wD}{}^E) = (R^*, 1)$ can be of two types: either a stable node or a stable focus. In what follows we first analyse the details of these two polar cases in the subsections III.4.1. and III.4.2. respectively, and then we propose two numerical examples in subsection III.4.3.

III.4.1. Stable node equilibrium

The following picture reproduces the phase-diagram of our economy for $\theta > \underline{\theta}$, where E is the stable node equilibrium (in Figure 1 we assume that $\theta > 1$).

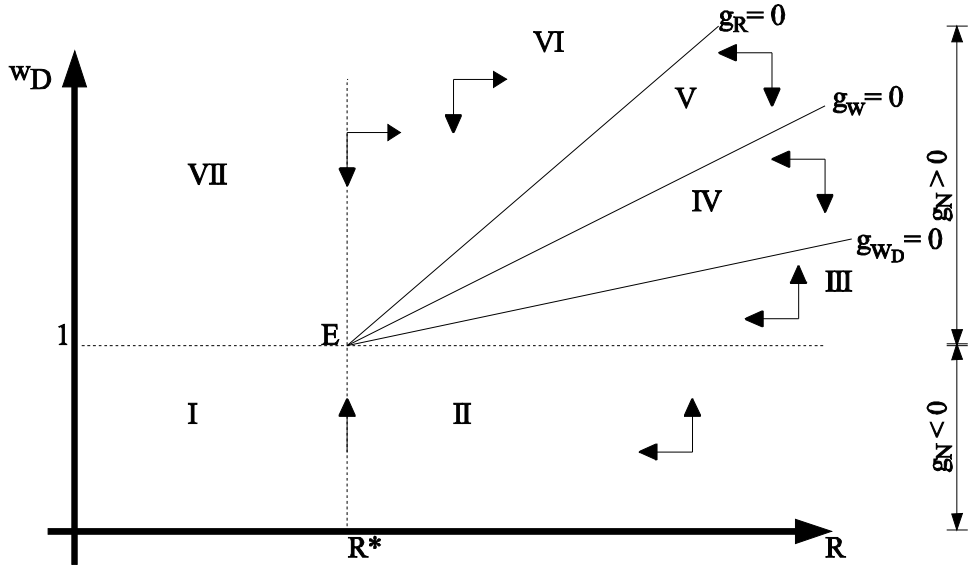

Figure 1: phase diagram of the stable node equilibrium

In Figure 1 we draw the loci where $g_R = 0$, $g_{wD} = 0$ and $g_w = 0$ in the space (R, w_D) and we partition the positive orthant in seven regions making the assumption that functions $\alpha(.)$, $\beta(.)$ and $\sigma(.)$ are linear.[22] Notice that the existence of region IV is the novelty in relation to the canonical Ricardian model, where the loci $g_{wD} = 0$ and $g_w = 0$ coincide.

Regions I and VII are not feasible because of the constraint on R. In region II $g_w > 0$, $g_{w^*} < 0$, $g_R < 0$, and $g_N < 0$, in region III $g_w > 0$, $g_{w^*} > 0$, $g_R < 0$, and $g_N > 0$, in region IV $g_w > 0$, $g_{w^*} > 0$, $g_R < 0$, and $g_N > 0$, in region V $g_w < 0$, $g_{w^*} > 0$, $g_R < 0$, and $g_N > 0$, and finally in region VI $g_w < 0$, $g_{w^*} > 0$, $g_R > 0$, and $g_N > 0$. The constraint on R works as a barrier, which is absorbing for $w_D < 1$ and repelling for $w_D > 1$. The overall dynamics indicates a convergence toward E, *i.e.* E is globally stable.[23]

To have a look at overall dynamics in Figure 2 we reproduce a numerical simulation of the economy.[24]

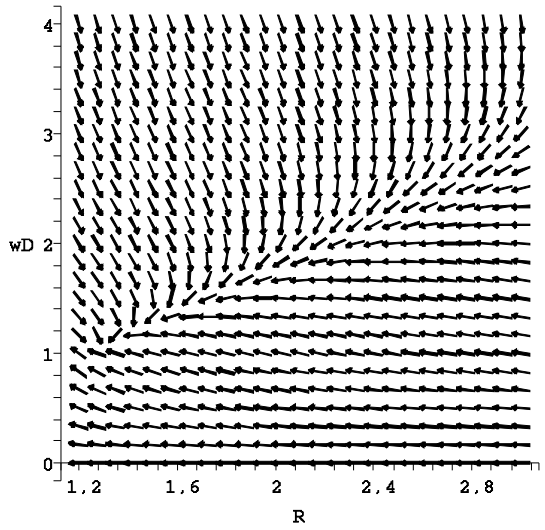

Figure 2: vector field of the stable node equilibrium

The equilibrium E in Figure 2 is represented by $(R^*, w_D^*) = (1.2, 1)$. We note that there is no cycle around E; but the convergence to equilibrium can include a non-linear dynamics both for R and w_D.

The model shows just one globally stable equilibrium in the space (R, w_D); but multiple steady-state equilibria in terms of the market wage, the natural wage and the labour force. In fact, given the same initial levels of the market wage, the rate of profits and the labour force, the higher is the initial natural wage, the higher is the wage in the long-run equilibrium and the lower the level of the labour force.[25] A heuristic argument for such findings is the following. Consider two economies with the same rates of profits, market wages and labour forces, but whose initial natural wages are below market wages but different. In particular, the first economy has a natural wage such that (R^1, w_D^1) is a point over the $g_W = 0$ curve in Figure 1, *i.e.* in region V; while the second economy has a higher natural wage, such that (R^2, w_D^2) is below the $g_W = 0$ curve, *i.e.* in region IV. Accordingly, the first economy is characterized by an initial decrease in w; while the second economy is characterized by an increase in w.

This different behaviour will lead the second economy to a higher equilibrium wage and a lower labour force (see (6)). Such findings support the claim made by classical economists according to which an economy where workers have finer tastes will show higher equilibrium wage. Say it differently, there exists hysteresis in the dynamics of the economy.[26] Figure 3 reproduces a numerical example of the trajectories of the market wage for the two economies, the one starting from region V and the other from region IV, which differ only for the initial level of their natural wage (the initial level of market wage is 0.769):[27]

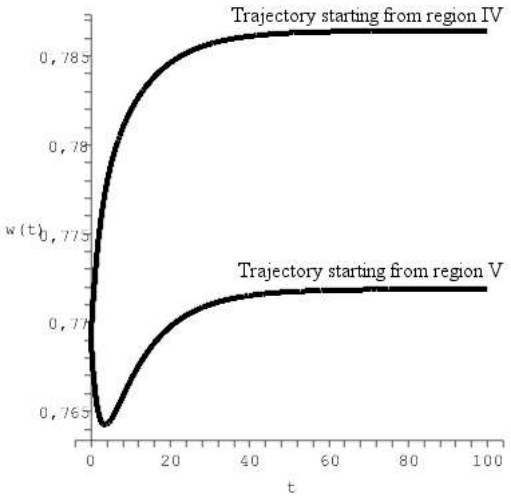

Figure 3: the effect on the equilibrium wage of different initial natural wages

The multiplicity of equilibria has relevant implications for empirical analysis as well. Consider two economies where w and N are increasing while R is decreasing. The first economy has a higher w*, such that it stays in region III, while the second economy stays in region IV. Since we can observe directly only R, w and N, but not w* the two economies are not distinguishable. However, they will have different equilibrium wages and they can show different dynamics to equilibrium. Figure 4 reproduces two trajectories of the market wage for the two economies, the one starting from region III and the other from region IV, which differs only for the initial level of their natural wage (a magnitude which, we stress again, is not directly observable):[28]

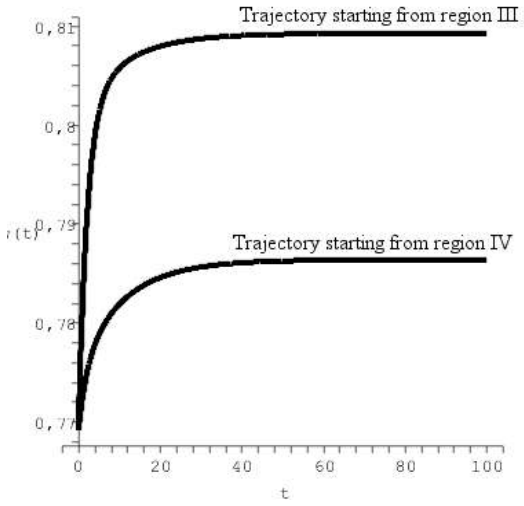

Figure 4: different long-run behaviours of two economies which are not observationally different

The same differences are present in the equilibrium level of labour.

III.4.2. Stable focus equilibrium

The dynamics for $\theta < \underline{\theta}$ is qualitative different. Again under the assumption that functions $\alpha(.)$, $\beta(.)$ and $\sigma(.)$ are linear, the locus $g_R = 0$ has negative slope. We know that E is locally stable, in particular a stable focus, therefore we expect possible overshooting in the dynamic paths of R and w_D near the equilibrium E. It is possible to prove that E is globally stable as in the previous case.[29] We notice that there also exists another equilibrium (point A in Figure 5), but it is unstable. Figure 5 reproduces the phase-diagram for this case.

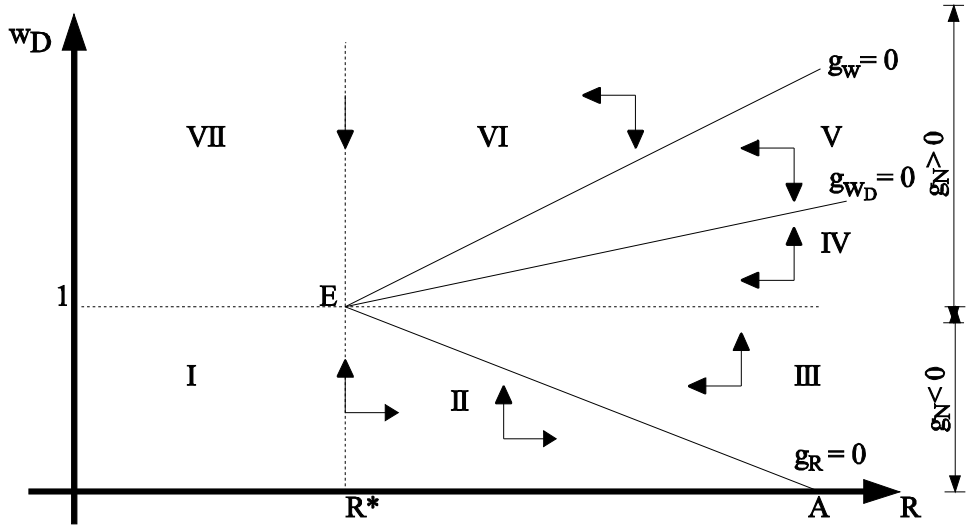

Figure 5: phase diagram of the stable focus equilibrium

Dynamics in the various regions can be easily calculated from the figure. In particular, consider the interesting behaviour of the system in region II. In the latter both the market rate of wages and the rate of profits are increasing, even if the rate of profits is over its long-run equilibrium level. However, when the economy enters into region II, only the market rate of wages continues to increase, while the rate of profit starts decreasing. Also in this case we have a multiplicity of equilibria in terms of the rate of wages and the level of labour force, so that the very same considerations made in subsection III.4.1 apply.

Figure 6 reproduces a numerical simulation, where the possibility of a cycle around the equilibrium is made manifest.[30]

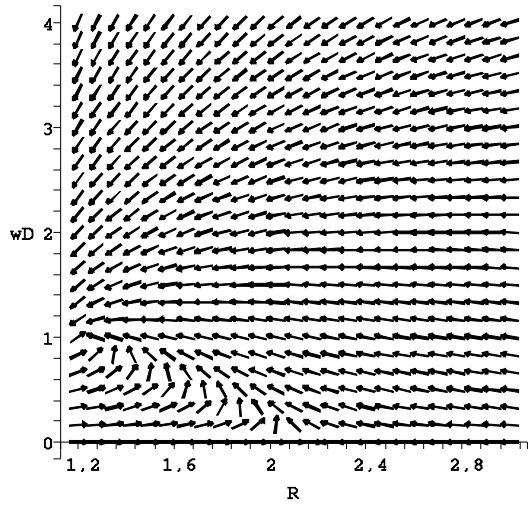

Figure 6: vector field of the stable focus equilibrium

A final remark concerns the relationship between the market rate and the natural rate of wages during the transition to the equilibrium, a point which has been lively debated in the literature. In general, we find that if the market wage is higher than the natural wage, then the two rates will become equal only in equilibrium, while if the market wage is lower than the natural wage then the former tends to become higher. Therefore, it is not possible for a market wage higher than the natural wage to become lower than the latter. However, this conclusion crucially depends on the assumption of the existence of a lower constraint on the value of the rate of profits. In fact, without the constraint on R, we could have a cycle around E if equilibrium E were a focus. Therefore w_D could be oscillating around 1 before the economy converges to E. This can happen in Hicks and Hollander's (1977) model, where R has to be only not negative.

III.4.3. Some numerical examples

In what follows we present two numerical examples. The first example concerns the stable node equilibrium. In particular, we consider an economy starting from region VI of Figure 1. Figure 7 reproduces a numerical simulation with the same parameters of Figure 2, where the initial levels

of the labour force, the natural wage and the rate of profits are such that the economy is starting in region VI (in particular the initial conditions are (w(0),R(0)) = (0.7692, 1.3)).

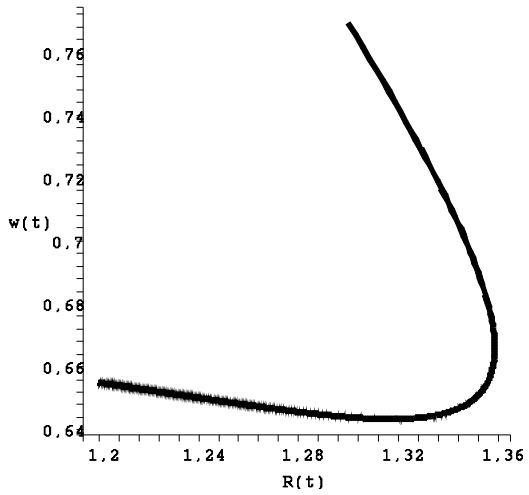

Figure 7: example of trajectory for the stable node equilibrium

Figure 7 shows that in the initial periods the economy performs a decrease in the market wage and an increase in R (economy is in region VI with respect to Figure 1). When the economy moves to region V, then R starts decreasing as well as the market wage. Then, the economy enters into region IV, where the market wage starts increasing, while R continues to decrease. Finally, the economy reaches the equilibrium, where R is equal to R* = 1.20 and w = w* = 0.66.

The other example regards the case where the equilibrium E is a stable focus. In Figure 8 we reproduces a numerical simulation for an economy starting from region II (the initial conditions are (w(0), R(0)) = (0.7692, 1.3)).

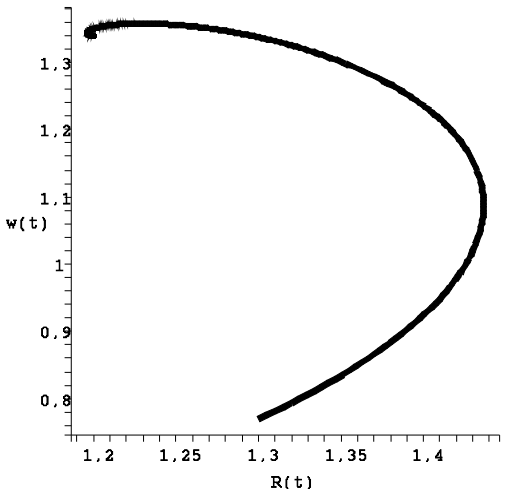

Figure 8: example of trajectory for the stable focus equilibrium

We observe that in the initial periods the market wage and the rate of profits are both increasing. When the economy reaches first region III and then region IV, R starts decreasing, while w is always increasing. Finally, the economy arrives at equilibrium R* = 1.20 and w = w* = 0.134. Both numerical simulations show that R and w have non linear dynamics. This is particularly interesting for the comparative static analysis, because the final outcome could be very different from the short-run behaviour of the economy.

IV. Final remarks

Ricardian growth models are generally built on the (often tacit) assumption of a constant natural wage (in real or commodity terms). Such an assumption, useful as it is in deriving interesting growth results, hides from sight that classical economists were perfectly aware of the fact that in growing economies (such as England) workers' normal patterns of consumption steadily rise both from a quantitative and a qualitative point of view; while in stagnant or declining economies (such as Ireland or China) workers' life keeps brutish and short.

In the first part of the paper we have gathered some classical hints on the relationship between economic growth and natural real wages in order to provide a rational reconstruction of the classical

point of view on natural wage dynamics. We have claimed that, within the classical framework, economic growth influences the dynamics of the natural real wage through workers' reactions to the wider consumption opportunities disclosed to them by the growth process. While in a stagnant economy the economic possibility to escape from the Malthusian poverty trap is almost *nihil*, in a growing economy workers earn 'high' real market wage and thus acquire the economic possibility to buy non-subsistence commodities. Moreover, workers become increasingly aware of the trade-off between children to rear and "comforts of life" to enjoy, and thus revise their fertility decisions in the light of rational economic reasonings as to their future standard of life. In such a scenario a 'high' real market wage does not provoke an increase of labouring population (as is the case in the Malthusian poverty trap scenario) but an upward revision of the "habits and customs " which shape natural real wages.

In the second part of the paper we have proposed a formal analysis. We have analysed the implication of a ricardian model with endogenous natural wage. The economy shows a globally stable dynamics, but the long-run equilibrium depends on the initial conditions: an economy with a higher initial natural wage show a higher equilibrium market wage and a lower labour force. Furthermore, the convergence to the long-run equilibrium can occur by cycles in the market wage, in the rate of profits and in the labour force. Thus short-run behaviour can be misleading in order to individuate the long-run equilibrium. Finally, since the natural wages is not observable, economies which appears to be initially identical can show a very different behaviour in the long-run.

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In what follows quotations from Adam Smith's *Wealth of Nations* refer to the Glasgow edition and are in the form (*WN* I.xi.5) indicating the Book, the chapter and the paragraph, respectively. Quotations from david Ricardo's *Principles of Political Economy and Taxation* refer to the Cambridge edition and are in the form (*Works* I.v.93) indicating the Volume, the chapter and the page number respectively.

[2] Modern scholars' interest in Ricardo's economics has been rejuvenated by Piero Sraffa's long-awaited edition of *The Works and Correspondence of David Ricardo* in the early 1950s and by Sraffa's 1960 provocative book, *Production of Commodities by Means of Commodities*. Peach (1993, Chapter I) and Klingen (1998) provide an overview of contemporary doctrinal debate on Ricardo. Caravale (1985) brings together some of the relevant contributions. See also Hollander (1995, Part IV) on Ricardian growth model(s) and the debate on Ricardo's theory of wages. For a book-length discussion of the classical theory of growth see Eltis (2000).

[3] Ricardian scholars generally agree that the market wage and the natural wage coincide in the stationary state when the growth both of capital and labouring population has come to a halt. It hardly needs to be stressed that classical economists look at the stationary state just as a mere possibility in the very far future for most known countries. The only notable exception is China, which, for Smith, has already "acquired that full complement of riches which the nature of its laws and institutions permits it to acquire" (*WN* I.viii.24).

[4] We are of course aware that the theoretical object called 'classical economics' is not a monolith, due to the many differences between the various classical authors, and that rival interpretations exist as to the nature and scope of 'classical economics'. Yet, we think that, differences of emphasis apart, classical authors substantially share a common point of view on the relationship between economic growth and the dynamics of the natural wage (in real or commodity terms).

[5] Smith defines the subsistence wage as the "lowest [rate of wages] which is consistent with common humanity" (*WN* I.vii.16). Ricardo claims that the natural rate of wages depends on the "quantity of food, necessaries, and conveniences become essential [to workers] from habit" (*Works* I.v.93). Hence, the normal basket of wage-goods "essentially depends on the habits and customs of the people" (*Works* I.v.96-97). Stirati (1994, 1995 and 1998) analyses the role played by notions of fairness and power relationships within the classical theory of natural wage determination.

[6] Malthus maintains that "in all cases where particular modes of subsistence, from whatever causes arising, have been for any time established, though such modes always remain susceptible of change, the change must be a work of time and difficulty" (Malthus 1986, vol. V, pp. 187-188).

[7] To make just an example, see Smith's detailed discussion of the different patterns of consumption and fertility of workers living in countries situated in different stages of economic development such as England, Scotland, North America, China, Bengal and some of the English settlements in the East Indies (*WN* I.viii.22 ff).

[8] The market price of labour is defined by Ricardo as "the price which is really paid for [labour], from the natural operation of the proportion of the supply to the demand" (*Works* I.v.94). Similarly, for Malthus, the market price of labour is "the actual price in the market, which from temporary causes is sometimes above, and sometimes below,

what is necessary to supply [the effectual demand of labourers]" (Malthus 1986, vol. V, p. 182).

[9] Smith defines the natural rate of wages as "the ordinary or average rate of wages" in a given time and place. He establishes a connection between the natural wage and economic development in so far as "[the circumstances which naturally determine the rate of wages] are affected by the riches or poverty, by the advancing, stationary or declining state of the society" (*WN* I.vii.34, see also WN I.vii.1). The state of economic development of a given country determines workers' normal consumption basket and their ability *and* economic convenience to rear a family (see *WN* I.viii.22 ff). Ricardo emphasises the relationship between the natural wage and subsistence (in the broad sense of the term) by claiming that the natural rate of wages is associated with a stationary labouring population (see *Works* I.v.93). Ricardo's notion of the natural wage is harshly criticized by Malthus. In Malthus' opinion, a stationary labouring population, usually associated to a stagnant economic situation, is a state of affairs "at a great distance in point of time". For Malthus, most of known economies have plenty of growth opportunities as yet unexploited. He prefers to define the "natural or necessary price of labour" as a long-run equilibrium price, that is, as "that price which, in the actual circumstances of the society, is necessary to occasion an average supply of labourers, sufficient to meet the effectual demand" (Malthus 1986, vol. V, p. 182).

[10] Smith claims that the "real recompense of labour, the real quantity of the necessaries and conveniences of life which it can procure to the labourer, has, during the course of the present century, increased perhaps in a still greater proportion than its money price. […] The common complaint that luxury extends itself even to the lowest ranks of the people, and that *the labouring poor will not now be contented with the same food, cloathing and lodging which satisfied them in former times*, may convince us that it is not the money price of labour only, but its real recompense, which has augumented" (*WN* I.viii.34, emphasis added). Ricardo plainly admits that "many of the conveniences now enjoyed in an English cottage, would have been thought luxuries at an earlier period of our history" (*Works* I.v.97). On the relationship between workers' fertility decisions and workers' taste for higher-quality commodities he maintains that the "friends of humanity cannot but wish that in all countries the labouring classes should have a taste for comforts and enjoyments, and that they should be stimulated by all legal means in their exertions to procure them. *There cannot be a better security against a superabundant population*. In those countries, where the labouring classes have the fewest wants, and are contented with the cheapest food, the people are exposed to the greatest vicissitudes and miseries" (*Works* I.v.100-101, emphasis added). Similarly, Malthus writes that "from 1720 to 1750 the price of wheat [in England] had so fallen, while [money] wages had risen, that instead of two thirds the labourer could purchase the whole of a peck of wheat with a day's labour". He adds that "the result was, that *their* [of the lower classes of people] *increased corn wages, instead of occasioning an increase of population exclusively, were so expended as to occasion a decided elevation in the standard of their comforts and conveniences*" (Malthus 1986, vol. V, pp. 185-186, emphasis added).

[11] As stressed in the Introduction, the market and the natural wage always coincide if, according to Samuelson's 1978 canonical classical model, the value of parameter ε is zero. By contrast, the market wage may be persistently above the natural wage if the value of parameter ε is positive.

[12] More on this point in Fiaschi and Signorino (2003a and b).

[13] Remind that, unlike the market wage, the natural wage is not a magnitude which may be directly observed. According to Ricardo's definition of the natural wage, the only way to ascertain whether, in a given time and place, the market wage and the natural wage diverge or coincide is to observe the related dynamics of population. Thus, *e vi termini*, any market rate of wages associated to a stationary population must be considered as a natural rate of wages.

[14] If W and N are assumed as given, Ricardo's theory of wages is interpreted as belonging to the wage-fund approach. By contrast, if W and w are assumed as given, Ricardo's theory of wages is interpreted as belonging to the natural wage or fix-wage approach. In the first case, eq. 4) determines the full-employment rate of wages; in the second case it determines the level of employment compatible with the state of capital accumulation (since K = W) and the ruling rate of natural wages.

[15] Since K = wN and wR = f'(N) we have that K = Nf'(N)/R: there exists an inverse relationship between K and r, given N and f'(.).

[16] Hick and Hollander (1977) write that such elasticity should be decreasing in N; in particular, it should be ∞ for N being equal to 0 and 0 for N being very large (but less than ∞). For example, the production function $f(N) = aN - cN^3$ for $N \in [0,(a/c)^{(1/2)}]$ shows these properties.

[17] A different equilibrium could exist, depending on the assumptions on the shape of $\alpha(.)$, $\beta(.)$, $\gamma(.)$ and $\theta(.)$, characterized by $w_D=0$. Notice that the trivial equilibrium $R = w_D = 0$ is not feasible.

[18] We ignore the constraint $R \geq R^*$ since it is not relevant in our discussion.

[19] It is straightforward to verify that if the roots are real then both are negative, while if they are complex, then the real part is negative.

[20] This assumption can have important effects on the overall dynamics, but we think it is not decisive for the study of local stability of equilibrium.

[21] A similar condition for the model of Hicks and Hollander (1977) is found by Gordon (1983).

[22] The linear representation of these loci is a simplifying assumption, depending on the relationships between the first derivatives of $\alpha(.)$, $\beta(.)$ and $\sigma(.)$. If all these function were non linear, then we exactly get the same picture from a qualitative point of view. In fact, given the constraint $R \gneq R^*$, we have that $(1 - 1/\theta)\,\alpha(w_D - 1) = \beta(R - R^*)$ is the locus $g_R = 0$ and its slope is given by $\beta'(R-R^*)/((1 - 1/\theta)\,\alpha'(w_D - 1))$ (> 0 for $\theta > 1$). In the same manner $\sigma(w_D - 1) = \beta\,(R - R^*)$ is the locus $g_{wD} = 0$ and its slope is given by $\beta'(R-R^*)/\sigma'(w_D - 1)$ (> 0 always) and $\alpha(w_D - 1) = \beta\,(R - R^*)$ is the locus $g_w = 0$ and its slope is given by $\beta'(R-R^*)/\alpha'(w_D - 1)$ (> 0 always, and greater than the slope of locus $g_{wD} = 0$). It is straightforward to check the relationships between the slopes of the different loci given the positive sign of all derivatives and since $\sigma(.) > \alpha(.)$ for $w_D > 1$.

[23] This statement can be rigorously proved observing that there exists a region where any possible trajectory starting from it cannot leave the region. This region can be easily found by taking an appropriate $R^M > R(0) > R^*$ and $w_D^M > w_D(0) > 1$, and considering the resulting compact set formed by any couple (R, w_D) such that $R \in [R^*, R^M]$ and $w_D \in [1, w_D^M]$). This region is positively invariant (see Hirsch and Smale (1974), p. 264). Moreover, the region contains only a singular point, the locally stable equilibrium E (this is straightforward from the figure). Finally, it is possible to show that $d(dR/dt)dR + d(dw_D/dt)dR < 0$ in every point of the plane, so that in the region a closed path, *i.e.* a limit cycle, cannot exist (see Bendixson's negative criterion in Gandolfo 1997, p. 438). The application of the Poincaré-Bendixson Theorem completes the proof (see Theorem 1 in Hirsch and Smale 1974, p. 251). We stress that this proof holds also when R has as lower bound 0 (instead of R*).

[24] In the simulation we assume that every function is linear and we use the following values $\beta = 0.2$, $R^* = 1.2$, $\alpha = 0.2$, and $\theta = 3$.

[25] Since $wR = f'(N)$, then in equilibrium, the higher is w the lower is N, given that R = R*.

[26] A suggestive interpretation is that the natural wage is the minimum level of wage which individuals believe to be necessary to have a "satisfying" life. Then the selection of equilibrium depends on this expected level of wage. In other words, the higher the individual expectations on the level of welfare the higher the equilibrium wage and therefore also the actual level of welfare. This phenomenon is usually called self-fulfilling expectations.

[27] Parameters'values are the same used for Figure 2, but initial natural wage, that for an economy is set to 1.08 (it is in region V) and for the other to 1.12 (it is in region IV).

[28] Parameters'values are the same used for Figure 3, but initial natural wage, that for an economy is set to 1.08 (it is in region IV) and for the other to 1.02 (it is in region III).

[29] The global stability of equilibrium E can be proved with the same reasoning of endnote 24, with the difference that R^M has to be always greater than the level of R corresponding to point A.

[30] In the simulation we again assume that every function is linear and used the following values $\beta = 0.2$, $R^* = 1.2$, $\alpha = 0.2$, and $\theta = 0.5$.